line in-plane swing angle and the platform pitch angle are calculated to be $-\alpha_{eq} = \theta_{eq} = -0.0372$ rad.

Figures 2 and 3 show variations of the real part of the least-damped oscillatory mode for both subsystems, with the Q and R matrices. Only by trial and error can one arrive at suitable values for Q and R that result in the desired closed-loop system response. The crosses show the optimal design point, arrived at after a series of parametric studies.

Figures 4 and 5 show the transient response of the differential length, the platform pitch, and the tether line in-plane swing angle, as well as platform yaw, roll, and tether line out-of-plane swing angle for initial conditions of 105 m in tether length, length rate 5.5×10^{-3} m/s and 0.05 rad in all the variational angles and angular rates of 5.5×10^{-5} rad/s. It is seen that it takes much more time for the out-of-plane subsystem to reach equilibrium than for the in-plane subsystem.

Concluding Remarks

The equations describing the out-of-plane motion (i.e., platform roll, yaw rotation, and tether out-of-plane swing) decouple from the in-plane motion equations (i.e., platform pitch rotation, tether in-plane swing, and motion in the direction of the tether) when the attachment point is offset only along the platform roll axis. The system is controllable with a momentum-type controller on the platform and with tension modulation on the tether line. The system is observable with tether length, length rate, platform roll and/or yaw angle, and their rate measurements. The tether attachment offset, which is the source of the system's natural coupling, is an important factor in establishing system controllability and observability. For the case of no attachment offset, rotation of the platform will not affect the subsatellite out-of-plane swing; in other words, we should consider the effect of higher-order terms or should augment the means of control, such as by placing actuators on the subsatellite to control the tether line out-of-plane swing.

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Vector Representation of Finite Rotations

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IN the following Note we restrict ourselves to real 3-vectors, right-handed Cartesian coordinate systems, and proper orthogonal transformation matrices. Euler showed that any sequence of rotations leading from one frame of reference to another can be represented by a single rotation about an axis fixed in space. The axis and sense of the rotation can be represented by a unit vector. It is a vector because the rotation axis exists independently of the coordinate systems chosen to represent the frames of reference. The angle of the rotation, like all angles, is a scalar. The product of a vector and a scalar is a vector. Therefore, the product of the unit vector and the angle of the rotation gives a vector that represents the rotation. This rotation vector can be shown to satisfy the definition of a vector: an ordered triple of real numbers that transforms from one coordinate system to another by an orthogonal transformation.

In 1949 Laning[1] derived the algebra and calculus of such rotation vectors. That is, he found the vector operation that corresponds to the combination of two rotations into a single equivalent rotation, and the vector differential equation that corresponds to the evolution of a rotation as a function of time, given the angular velocity of one frame of reference with respect to another.

In 1950 Goldstein[2] wrote:

> Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are two such [rotation] "vectors" associated with transformations A and B. Then to qualify as vectors they must be commutative in addition...But the addition of two rotations, i.e., one rotation performed after another, it has been seen, corresponds to the product AB of the two matrices. However, matrix multiplication is not commutative...hence $\boldsymbol{A}$, $\boldsymbol{B}$ are not commutative in addition and cannot be accepted as vectors.

In 1965 Greenwood[3] wrote: "...one might think that a single rotation $\boldsymbol{\phi} = \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2$ is equivalent to the rotations $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ taken in sequence. But we have seen that rigid body rotations are not commutative, in general, whereas vector addition is commutative. Therefore a general rigid-body rotation is not a vector quantity."

In 1970 Meirovitch[4] wrote: "...in general, matrix products are not commutative...Hence, *finite angles of rotation cannot be represented by vectors.*"

In 1986 Hughes[5] wrote: "because rotation matrices do not commute in multiplication, the search for an angular displacement 'vector' is futile."

These authors use the following reasoning: the combination of rotations does not commute; the addition of vectors commutes; therefore rotations cannot be represented by vectors. Of course, the correct conclusion is that the combination of rotations cannot be represented by the addition of vectors. (The addition of two rotation vectors does, in fact, commute, but it does not, in general, represent the combination of the two corresponding rotations. Laning's vector operation does not commute and does represent the combination of the two rotations.)

It is pedagogically important to understand that whether or not an ordered triple of real numbers is a vector is determined by its transformation properties, not by whether particular vector operations on it are physically meaningful. The operations follow from the definition of a vector and the laws of real number arithmetic, not vice versa. The operations we use are selected for their physical usefulness from the set of all possible operations on vectors. Whenever we apply vectors to a new analysis, we may need to invent a new vector operation as Laning did.

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